Numerical-Valued Polynomials

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Background: Monoids

Throughout this talk, a *monoid* M is a semigroup ~~with identity (which we denote by 1)~~ that is cancellative and commutative.

Ex: $(\mathbb{N}, \times)$, $(\mathbb{N}_0, +)$

We denote by $M^\times$ the set of invertible elements of M. We say that M is a group if $M^\times = M$, while M is *reduced* if $|M^\times| \leq 1$. We use M_{red} to denote the quotient $M/M^\times$, which is also a monoid ("and $M_{\text{red}} = M$").

Given a monoid M, we denote by $\mathcal{G}(M)$ the *quotient group* of M.

Ex: $(\mathbb{Q}_{>0}, \times)$, $(\mathbb{Z}, +)$

An element $a \in M \setminus M^\times$ is called an atom if $a = b \cdot c$ for some $b, c \in M$ implies that either $b \in M^\times$ or $c \in M^\times$. We denote by $\mathcal{A}(M)$ the set of atoms (or irreducibles) of M. A monoid M is atomic provided that every $b \in M \setminus M^\times$ can be expressed as a (finite) product of atoms.

Background: Factorizations

Let M be an atomic monoid.

- $\mathsf{Z}(M)$ denotes the free commutative monoid generated by the set of atoms $\mathcal{A}(M_{\text{red}})$.
- There is a unique monoid homomorphism $\pi : \mathsf{Z}(M) \to M_{\text{red}}$ fixing all atoms.
- The length ℓ of a factorization $z = a_1 \cdots a_\ell$, where $a_1, \ldots, a_\ell \in \mathcal{A}(M)$, is denoted by $|z|$.
- For each $b \in M$, we define the sets

$$\mathsf{Z}_M(b) = \pi^{-1}(bM^\times) \subseteq \mathsf{Z}(M) \quad \text{and} \quad \mathsf{L}_M(b) = \{|z| : z \in \mathsf{Z}_M(b)\} \subseteq \mathbb{N}_0.$$

- The monoid M is a finite factorization monoid (FFM) if $\mathsf{Z}(b)$ is finite for all $b \in M$.
- The monoid M is a unique factorization monoid (UFM) if $\mathsf{Z}(b)$ is a singleton for all $b \in M$.

Background: Semirings

A *commutative semiring* S is a nonempty set endowed with two binary operations denoted by '$+$' and '$\cdot$' and called *addition* and *multiplication*, respectively, such that the following conditions hold:

- $(S, +)$ is a monoid with an identity element denoted by 0;
- $(S, \cdot)$ is a commutative semigroup ~~with an identity element denoted by 1~~;
- $b \cdot (c + d) = b \cdot c + b \cdot d$ for all $b, c, d \in S$;
- $0 \cdot b = 0$ for all $b \in S$.

A subset S' of a semiring S is a *subsemiring* of S if $(S', +)$ is a submonoid of $(S, +)$ that contains 0 and is closed under multiplication.

Background: Semidomains

Definition

We say that a semiring S is a *semidomain* provided that S is a subsemiring of an integral domain.

Examples: integral domains, $\mathbb{N}_0$, $\mathbb{N}_0[x]$, $\mathbb{N}_0[\![x]\!]$, *positive semirings* (i.e., subsemirings of $\mathbb{R}_{\geq 0}$)

Lemma (Gotti and P., 2023)

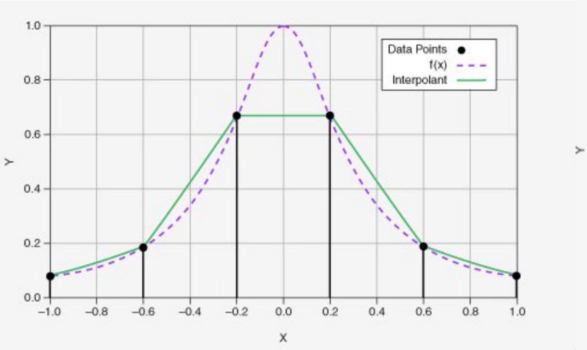
For a semiring S (with multiplicative identity), the following conditions are equivalent.

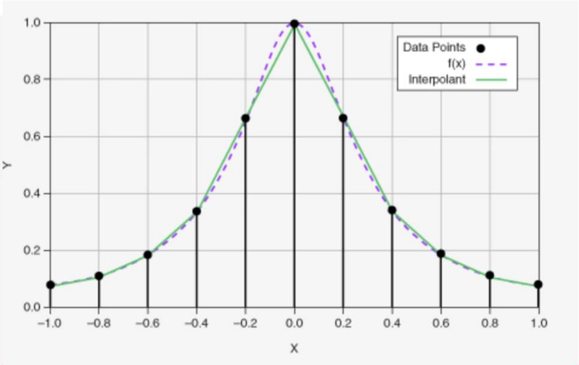
(a) S is a semidomain.

(b) The multiplication of S extends to $\mathcal{G}(S)$ turning $\mathcal{G}(S)$ into an integral domain.

Motivation

Linear Interpolation





Instead of using lines, let's use... (drumroll please)...

$$\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}.$$

Motivation

Set $\mathrm{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$. A few observations about $\mathrm{Int}(\mathbb{Z})$:

1. $\mathrm{Int}(\mathbb{Z})$ is an integral domain;
2. $\mathbb{Z}[X] \subsetneq \mathrm{Int}(\mathbb{Z}) \subsetneq \mathbb{Q}[X]$;
3. $\mathrm{Int}(\mathbb{Z})$ is not Noetherian (recall that a ring is Noetherian if every ideal is finitely generated);
4. For f to be in $\mathrm{Int}(\mathbb{Z})$ it is enough to verify that $f(\mathbb{N}) \subseteq \mathbb{Z}$;
5. $\left\{\binom{X}{n} \;\middle|\; n \in \mathbb{N}_0\right\}$ is a basis of $\mathrm{Int}(\mathbb{Z})$ (as a $\mathbb{Z}$-module).

Motivation

A numerical monoid is a cofinite submonoid of $(\mathbb{N}_0, +)$.
Ex: $\{0, 2, 3, 4, 5, \ldots\}$, $\{0, 3, 5, 6, 8, 9, 10, \ldots\}$

A few facts about numerical monoids:

1. Numerical monoids are finitely generated;
2. Numerical monoids are FFMs;
3. Numerical monoids are closed under multiplication (so they are semirings!);
4. Numerical monoids are fun!

Numerical-Valued Polynomials

Definition

Let N be a numerical monoid. We set $\operatorname{Int}(N) := \{f \in \mathbb{Q}[X] \mid f(N) \subseteq N\}$. The elements of $\operatorname{Int}(N)$ are called numerical-valued polynomials.

A few observations about $\operatorname{Int}(N)$:

1. $N \subseteq N[X] \subseteq \operatorname{Int}(N) \subseteq \operatorname{Int}(\mathbb{Z})$;
2. $\operatorname{Int}(N)$ is a semidomain;
3. Numerical-valued polynomials seem super fun!

Natural-Valued Polynomials

Definition

We set $\mathrm{Int}(\mathbb{N}_0) := \{f \in \mathbb{Q}[X] \mid f(\mathbb{N}_0) \subseteq \mathbb{N}_0\}$. The elements of $\mathrm{Int}(\mathbb{N}_0)$ are called natural-valued polynomials .

$$\left\{\sum_{i=1}^{n} k_i \binom{X}{n_i} \mid k_i \in \mathbb{N}\right\} \subsetneq \mathrm{Int}(\mathbb{N}_0) \subsetneq \mathrm{Int}(\mathbb{Z}).$$

Examples

Observe that $\binom{X}{2} > \binom{X}{1}$ for all positive integers bigger than 3. Hence the polynomial $3\binom{X}{2} - \binom{X}{1} \in \mathrm{Int}(\mathbb{N}_0)$. We can also conclude that $\mathbb{N}_0[X] \subsetneq \mathrm{Int}(\mathbb{N}_0)$. On the other hand, the polynomial $3X - 4 \in \mathrm{Int}(\mathbb{Z}) \setminus \mathrm{Int}(\mathbb{N}_0)$.

Natural-Valued Polynomials

Proposition (P., 202?)

We have that $\mathcal{G}(\mathrm{Int}(\mathbb{N}_0)) = \mathrm{Int}(\mathbb{Z})$.

Proof: Let $f \in \mathrm{Int}(\mathbb{Z})$. We already established that we can write $f = \sum_{i=0}^{n} c_i \binom{X}{n_i}$, where $c_0, \ldots, c_n \in \mathbb{Z}$. Consequently,

$$f = \sum_{i=0}^{m} c_{t_i} \binom{X}{n_{t_i}} + \sum_{j=0}^{\ell} c_{r_j} \binom{X}{n_{r_k}},$$

where all coefficients are positive. Hence $\mathrm{Int}(\mathbb{Z}) \subseteq \mathcal{G}(\mathrm{Int}(\mathbb{N}_0))$. The reverse implication follows after noticing that if $f, g \in \mathrm{Int}(\mathbb{N}_0)$ then $(f - g)(\mathbb{N}) \subseteq \mathbb{Z}$. ∎

Natural-Valued Polynomials

Let S be a subset of $\mathbb{Z}$, and set $\mathrm{Int}(S,\mathbb{Z}) := \{f \in \mathbb{Q}[X] \mid f(S) \subseteq \mathbb{Z}\}$. Observe that $\mathrm{Int}(\mathbb{Z}) = \mathrm{Int}(\mathbb{Z},\mathbb{Z})$.

Question: Can we describe all subsets $S \subseteq \mathbb{Z}$ such that $\mathrm{Int}(S,\mathbb{Z}) = \mathrm{Int}(\mathbb{Z})$?

Yes, we can!

Follow-up question: Can we describe all subsets $S \subseteq \mathbb{N}_0$ such that $\mathrm{Int}(S,\mathbb{N}_0) = \mathrm{Int}(\mathbb{N}_0)$?

Proposition (P., 202?)

Let $S \subseteq \mathbb{N}_0$. Then $\mathrm{Int}(S,\mathbb{N}_0) = \mathrm{Int}(\mathbb{N}_0)$ if and only if $S = \mathbb{N}_0$.

Elasticity of Natural-Valued Polynomials

For an atomic monoid M, the elasticity of a nonunit $b \in M$, denoted by $\rho(b)$, is defined as

$$\rho(b) = \frac{\sup \mathsf{L}(b)}{\inf \mathsf{L}(b)}.$$

By convention, we set $\rho(u) = 1$ for every $u \in M^\times$. It is easy to see that, for all $b \in M$, we have that $\rho(b) \in \{\infty\} \cup \mathbb{Q}_{\geq 1}$. The elasticity of the monoid M is defined to be

$$\rho(M) := \sup\{\rho(b) \mid b \in M\}.$$

The set of elasticities of M is denoted by $R(M) := \{\rho(b) \mid b \in M\}$, and M is said to have full elasticity provided that, for $q \in \mathbb{Q} \cap [1, \rho(M)]$, there exists an element $b \in M \setminus M^\times$ such that $\rho(b) = q$.

Elasticity of Natural-Valued Polynomials

It is known that $\text{Int}(\mathbb{Z})$ has full and infinite elasticity (no way!).

Proposition (P., 202?)

The semidomain $\text{Int}(\mathbb{N}_0)$ has infinite elasticity.

Proof:

$$n(n-1)\binom{X}{n} = [x-(n-1)][x-(n-2)]\binom{X}{n-2}$$

Open questions

1. Is $\text{Int}(\mathbb{N}_0)$ fully elastic?
2. What are the irreducibles of $\text{Int}(\mathbb{N}_0)$?

References

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Thank you!